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Convergence of the Schwinger–DeWitt Expansion for Some Potentials

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The time dependence of the evolution operator kernel for the Schrödinger equation has been studied with a help of the Schwinger–DeWitt expansion. For many of potentials this expansion is divergent. But there are nontrivial potentials for which the Schwinger–DeWitt expansion is convergent. These are, e.g., $V = g/x^2$, $V = -g/\cosh^2 x$, $V = g/\sinh^2 x$, $V = g/\sin^2 x$. For all of them the expansion is convergent when $g = \lambda(\lambda - 1)/2$ and λ is integer. The theories with these potentials have no divergences and in this sense they are "good" potentials, in contrast to other ones. So it seems natural to pay special attention to these "good" potentials. Besides convergence, they have another interesting feature: convergence takes place only for discrete values of the charge g. Hence, in the theories of this class the charge is quantized.

1. GENERAL RELATIONS

Various approaches in quantum theory use the short-time Schwinger–DeWitt expansion (Schwinger, 1951; DeWitt, 1965, 1975; Osborn and Fujiwara, 1983; Barvinsky *et al.*, 1995). As with other expansions in different parameters—conventional perturbation theory (Bender and Wu, 1969; Lipatov, 1977), the WKB expansion, $1/n$ expansion (Popov *et al.*, 1992), etc.—it is usually treated as an asymptotic one. Because of the divergence of these expansions, many difficulties arise in quantum theory. In particular, in QCD one cannot obtain correct predictions for low-energy phenomena and so on.

Here we consider one interesting phenomenon related to the Schwinger–DeWitt expansion which may allow us in the future to deal with some problems arising from the divergence of the series.

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The Schwinger–DeWitt expansion is a specific representation of the solution of the following problem for the evolution operator kernel:

$$i\frac{\partial}{\partial t}\langle q', t|q, 0\rangle = -\frac{1}{2}\frac{\partial^2}{\partial q'^2}\langle q', t|q, 0\rangle + V(q')\langle q', t|q, 0\rangle \tag{1}$$

with initial condition

$$\langle q', t = 0|q, 0\rangle = \delta(q' - q) \tag{2}$$

The kernel $\langle q',t|q,0\rangle$ is written as

$$\langle q', t|q, 0\rangle = \frac{1}{\sqrt{2\pi it}}\exp\left\{i\frac{(q' - q)^2}{2t}\right\}F(t; q', q) \tag{3}$$

and F according to Schwinger (1951), DeWitt (1965, 1975), Slobodenÿuk (1993, 1995a) is expanded in powers of t,

$$F(t; q', q) = \sum_{n=0}^{\infty}(it)^n a_n(q', q) \tag{4}$$

Here and everywhere below, we use dimensionless values defined in an obvious manner. The potential $V(q)$ is continuous function.

It is easy to derive from (1)–(3) the problem for the function F. The latter should satisfy the equation

$$i\frac{\partial F}{\partial t} = -\frac{1}{2}\frac{\partial^2 F}{\partial q'^2} + \frac{q' - q}{it}\frac{\partial F}{\partial q'} + V(q')F \tag{5}$$

and initial condition

$$F(t = 0; q', q) = 1 \tag{6}$$

The coefficient functions $a_n(q', q)$ may be determined from the sequence of recurrent relations

$$a_0(q', q) = 1 \tag{7}$$

$$na_1(q', q) + (q' - q)\frac{\partial a_1(q', q)}{\partial q'} = a_1(q', q') = -V(q') \tag{8}$$

for $n > 1$

$$na_n(q', q) + (q' - q)\frac{\partial a_n(q', q)}{\partial q'} = \frac{1}{2}\frac{\partial^2 a_{n-1}(q', q)}{\partial q'^2} - V(q')a_{n-1}(q', q) \tag{9}$$

Equations (7)–(9) show that a_n can be calculated via a_{n-1} by means of integral relations

$$a_n(q', q) = \int_0^1 \eta^{n-1} d\eta \left\{ \frac{1}{2} \frac{\partial^2}{\partial x^2} - V(x) \right\} a_{n-1}(x, q) \bigg|_{x=q+(q'-q)\eta} \tag{10}$$

Combinations of equation (10) for different numbers n allow us to represent a_n for given n through the potential V and its derivatives

$$\begin{aligned} a_n(q', q) = &-\int_0^1 \eta_n^{n-1}\, d\eta_n \int_0^1 \eta_{n-1}^{n-2}\, d\eta_{n-1} \cdots \int_0^1 \eta_2\, d\eta_2 \int_0^1 d\eta_1 \\ &\times \left\{ \frac{1}{2} \frac{\partial^2}{\partial x_n^2} - V(x_n) \right\} \left\{ \frac{1}{2} \frac{\partial^2}{\partial x_{n-1}^2} - V(x_{n-1}) \right\} \\ &\times \cdots \left\{ \frac{1}{2} \frac{\partial^2}{\partial x_2^2} - V(x_2) \right\} V(x_1) \end{aligned} \tag{11}$$

Here $x_i = q + (x_{i+1} - q)\eta_i$, $x_{n+1} = q'$. Derivatives with respect to different x_i may be easily connected with each other,

$$\frac{\partial}{\partial x_i} = \eta_{i-1} \frac{\partial}{\partial x_{i-1}} = \eta_{i-1}\eta_{i-2} \frac{\partial}{\partial x_{i-2}} \tag{12}$$

etc.

Another useful representation for the kernel may be obtained if in some domain for the potential $V(q)$ the Taylor expansion is used

$$V(q') = \sum_{k=0}^{\infty} \Delta q^k \frac{V^{(k)}(q)}{k!} \tag{13}$$

where $V^{(k)}(q)$ is the kth derivative of $V(q)$, $\Delta q = q' - q$; then the following representation of F may be used in calculations:

$$F(t; q', q) = 1 + \sum_{u=1}^{\infty} \sum_{k=0}^{\infty} (it)^n \Delta q^k b_{nk}(q) \tag{14}$$

For the coefficients b_{nk} one has algebraic recurrence relations:

$$b_{1k} = -\frac{V^{(k)}(q)}{(k+1)!} \tag{15}$$

$$b_{nk} = \frac{1}{n+k} \left[\frac{(k+1)(k+2)}{2} b_{n-1,k+2} - \sum_{m=0}^{k} \frac{V^{(m)}(q)}{m!} b_{n-1,k-m} \right] \tag{16}$$

Both representations (11) and (14)–(16) are used for analysis of convergence of the Schwinger–DeWitt expansion (behavior of a_n for $n \to \infty$) in the general case and for specific potentials.

This formalism can be easily modified for application to singular potentials with singularity of type $1/q^2$ at $q = 0$ (Slobodenyuk, 1996a). One should take instead of initial condition (2) the following one:

$$\langle q', t = 0 | q, 0\rangle = \delta(q' - q) + A\delta(q' + q) \tag{17}$$

which may provide fulfillment of the boundary condition for the wave function $\psi(q)$ at $q = 0$ [$\psi(q)$ should vanish at $q = 0$] by appropriate choice of the constant A. The constant A is determined by the requirement that the kernel does not have a singularity at $q = 0$ or $q' = 0$ ($t \neq 0$). In correspondence with (17), the kernel is represented through two functions $F^{(\pm)}$ as

$$\langle q', t | q, 0\rangle = \frac{1}{\sqrt{2\pi i t}} \exp\left\{ i \frac{(q' - q)^2}{2t} \right\} F^{(-)}(t; q', q) + A \frac{1}{\sqrt{2\pi i t}} \exp\left\{ i \frac{(q' + q)^2}{2t} \right\} F^{(+)}(t; q', q) \tag{18}$$

where $F^{(\pm)}$ can be expanded analogously to (4).

Generalization to the three-dimensional case is obvious, so we will not discuss it specially.

2. EXAMPLES OF CONVERGENT EXPANSIONS

It was shown in Slobodenyuk (1995a) that $|a_n(q',q)| \sim \Gamma(bn)$, where $0 < b \leq 1$, if there are no cancellations between different contributions to a^n. For the most potentials such factorial growth really takes place. Namely, for $V(q)$ represented by a polynomial of order L, the constant b is given by $b = (L - 2)/(L + 2)$; for other $V(q)$ the constant b is equal to 1. So, the Schwinger–DeWitt expansion factorially diverges and the point $t = 0$ is essentially a singular point of the function F. But there exist some kinds of potentials for which cancellations are so essential that the series (4) is convergent, and F is analytic at the point $t = 0$. This takes place only for some discrete values of charge. Examples of such potentials were presented in Slobodenyuk (1995b, 1996.):

$$V(q) = \frac{q}{\cosh^2 q} \tag{19}$$

$$V(q) = \frac{g}{q^2} \tag{20}$$

$$V(q) = \frac{g}{\sinh^2 q} \tag{21}$$

$$V(q) = \frac{g}{\sin^2 q} \tag{22}$$

$$V(q) = aq^2 + g/q^2 \tag{23}$$

For all of them the expansion is convergent when $g = \lambda(\lambda - 1)/2$ and λ is integer.

For illustration we consider the potential (20). This potential has singularity at $q = 0$, so the special formalism should be applied here. But for the sake of brevity we calculate only the function $F^{(-)}$, which is denoted simply as F.

Expansion (13) for the potential (20) has the finite convergence range $R(q) = q$, the finiteness of which is connected with the singularity of $V(q)$ at the point $q = 0$. The derivatives $V^{(k)}$ may be easily calculated,

$$V^{(k)}(q) = (-1)^k \frac{\lambda(\lambda - 1)}{2} \frac{(k + 1)!}{q^{k+2}} \tag{24}$$

Substituting (24) into (15), (16) and reducing n times the first index of $b_{nk}^{(-)}$ by means of (16), we get

$$\begin{aligned} b_{nk}^{(-)} &= \frac{(-1)^{n+k}}{q^{2n+k}} \frac{(k + n - 1)!}{n!(n - 1)!k!} \prod_{j=1}^{n} \left(\frac{\lambda(\lambda - 1)}{2} - \frac{j(j - 1)}{2} \right) \\ &= \frac{(-1)^{n+k}}{q^{2n+k}} \frac{(k + n - 1)!}{n!(n - 1)!k!} \frac{\Gamma(\lambda + n)}{2^n \Gamma(\lambda - n)} \end{aligned} \tag{25}$$

It is obvious that if λ is noninteger, then $|b_{nk}^{(-)}| \sim n!$ for $n \to \infty$. So, for noninteger λ, the expansion (14) for potential the (20) is divergent. But if λ is integer ($\lambda > 1$, because the cases $\lambda = 0$, $\lambda = 1$ are trivial), then one can easily see from (25) that only the coefficients $b_{nk}^{(-)}$ for $n < \lambda$ are different from zero, and in (14) the series in powers of t is really a polynomial of finite degree $\lambda - 1$. Let us substitute (25) into (14) and make summation over k. Then we get finally

$$F^{(-)}(t; q', q) = 1 + \sum_{n=1}^{\infty} \left(\frac{-it}{2q'q} \right)^n \frac{\Gamma(\lambda + n)}{n! \Gamma(\lambda - n)} \tag{26}$$

The series in (26) has the following feature: if λ is noninteger, then the coefficient in front of t^n growths as $n!$ for $n \to \infty$ and the series is asymptotic; but if λ is integer, then the series contains only a finite number of terms (the sum is carried out to $n = \lambda - 1$). So, in the latter case the expansion is convergent.

Singularities of $F^{(\pm)}$ at $q = 0$ cancel each other and the kernel $\langle q', t | q, 0\rangle$ has no such singularity. Substitution of the obtained expressions for $F^{(\pm)}$ into (18) gives us the asymptotic expansion of the function

$$\langle q', t | q, 0\rangle = \exp\left[-i\frac{\pi}{2}\left(\lambda - \frac{1}{2}\right)\right]\frac{\sqrt{q'q}}{it} \times \exp\left\{i\frac{q'^2 + q^2}{2t}\right\} J_{\lambda-1/2}\left(\frac{q'q}{t}\right) \tag{27}$$

for small t (large $q'q/t$). Equation (27) coincides with the known expression derived by other methods. The asymptotic expansion for the Bessel function is not divergent only for semiinteger order, i.e., for integer λ. In this case the series contains a finite number of terms and the point $t = 0$ is regular point, in contrast to the case of noninteger λ, when $t = 0$ is essentially singular point.

For other potentials of the series (19)–(23) the situation is similar. The expansion (4) is convergent only for integer λ and divergent in other cases (but it includes an infinite number of terms).

Moreover, careful study shows us that if we consider the coupling constant g of continuous potential $V(q)$ as an independent variable, then the coefficients a_n of the representation (3)–(4) for the evolution operator kernel increase for $n \to \infty$ as

$$a_n \sim \Gamma\left(n\frac{L-2}{L+2}\right)$$

for the potentials expressed via the polynomial of order L, and as

$$a_n \sim n!$$

for other potentials. So, the Schwinger–DeWitt expansion in this supposition is divergent for all potentials, excluding polynomials of order not higher than two.

If the charge is treated as a fixed parameter, then because of cancellations, for some kinds of potentials and some values of the charge g, the expansion (4) may be convergent. Examples of such potentials have been presented above. Discreteness of the charge for the class of the potentials for which the Schwinger–DeWitt expansion is convergent may be connected to the discreteness of the charge in nature. In this correspondence, the potentials of this class are of special interest. Operating with them, we get rid of some kinds of divergences in the theory and at the same time have a theory with discrete charge. One should look for other potentials of this class and study them carefully. Besides, it is interesting to extend such analysis to quantum field theory. One may hope that it will allow us to reconstruct quantum electrodynamics with exactly fixed charge *e*.

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